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1995 J. Phys. A: Math. Gen. 28 1627

(http://iopscience.iop.org/0305-4470/28/6/017)

J. Phys. A: Math. Gen. 28 (1995) 1627–1640. Printed in the UK

Universal parametric correlations of eigenvalues of random matrix ensembles

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Received 19 October 1994, in final form 16 January 1995

Abstract. Eigenvalue correlations of random matrix ensembles as a function of an external (parametric) perturbation are investigated via the Dyson Brownian-motion model in the situation where the level density has a hard edge singularity. By solving a linearized hydrodynamical equation, a universal dependence of the density–density correlator on the external field is found. As an application, we obtain a formula for the variance of linear statistics with the parametric dependence exhibited as a Laplace transform.

1. Introduction

Eigenvalues of $N \times N$ matrices can be viewed as the energy levels (x_a, $a = 1, \ldots, N$) of an effective Hamiltonian

$$H_0 = \sum_a x_a n_a \tag{1}$$

where the n_a are the occupation at level a. A proposal by Wigner [1] is that the levels are drawn from an ensemble of matrices and, when restricted to the eigenvalues, have the joint probability distribution

$$P(x_1, \ldots, x_N) \prod_{a=1}^{N} \mathrm{d}x_a = C_N \exp[-\beta W(x_1, \ldots, x_N; u)] \prod_{a=1}^{N} \mathrm{d}x_a \tag{2}$$

where

$$W(x_1, \ldots, x_N; u) = -\sum_{a<b} \ln|x_a - x_b| + \sum_a u(x_a) \tag{3}$$

C_N is a constant and $\beta = 2, 1, 4$ describes ensembles with unitary, orthogonal and symplectic symmetries, respectively. The density of the levels is defined by

$$\sigma(x) := \left\langle \sum_a \delta(x - x_a) \right\rangle_{\mathrm{eq}} \tag{4}$$

where $\langle \cdots \rangle_{\mathrm{eq}}$ is an average with weight P of equation (2). For $u(x) = x^2$ and x_a supported on the real line, σ is given by the Wigner semi-circular law [1] in the limit of large N,

$$\sigma_{\mathrm{W}}(x) \sim \sqrt{N - x^2} \qquad x \in \left(-\sqrt{N}, \sqrt{N}\right). \tag{5}$$

Other ensembles arising from transport in disordered electronic systems [2] have x_a supported on the right half-line with different confining potentials, $u(x)$. For $u(x) = x^\alpha$ and $\alpha > \frac{1}{2}$ one finds the eigenvalue density [3,4]

$$\sigma(x) \sim \sqrt{\frac{N}{x}} \qquad x \ll N \tag{6}$$

is universal near the 'hard edge' [5], in the sense that it is independent of α. It can be shown that for $0 < \alpha < \frac{1}{2}$, $\sigma(x) \sim 1/x^{1-\alpha}$ [4]. In this paper we are interested in the response of the levels when H_0 is perturbed by an external potential. Of particular interest is the eigenvalue correlator as a function of the external potential. This problem was first studied in the context of the energy-eigenvalues distribution of a disordered metallic ring subjected to an external magnetic field, using diagrammatic techniques [6]†, and it was found that the eigenvalue correlations are universal after an appropriate rescaling. These results were later reproduced in [7] using the Brownian-motion model of Dyson [8] in the hydrodynamical approximation. Exact correlations for all strengths of the perturbation were obtained in [9] using the method of supersymmetry pioneered by Efetov [10]. All of the above results are valid in the bulk of the spectrum where the density is uniform, $\sigma_{\text{bulk}} = \sigma_{\text{W}}(0) = \text{constant}$. The random matrix ensembles investigated in [2] follows from the *global maximum entropy ansatz*. In this formulation the $1/\sqrt{x}$ singularity is a *generic* feature for a general class of confining potential as discussed above and in [3,4]. To interpret the time parameter as an external field would require a specific microscopic model. This was partially accomplished in [12].

The phenomenological theory proposed by Dyson [8] interprets the eigenvalues x_a as positions of classical particles which are governed by an over-damped Langevin equation subjected to a Gaussian random force $f_a(\tau)$,

$$\gamma \frac{\mathrm{d}x_a}{\mathrm{d}\tau} = -\frac{\partial W}{\partial x_a} + f_a(\tau) \tag{7}$$

where γ is the friction coefficient,

$$\overline{f_a(\tau)} = 0 \qquad \overline{f_a(\tau) f_b(\tau')} = \frac{2\gamma}{\beta} \delta_{ab} \delta(\tau - \tau') \tag{8}$$

and τ is related to the strength of the perturbation, X. Since the x_a's undergo a Brownian motion it is to be expected that $X^2 \propto \tau$ [8, 11]. A Fokker–Planck equation that describes the 'time'-dependent joint probability distribution can be derived, and reads

$$\gamma \frac{\partial}{\partial \tau} P(x_1, \ldots, x_N, \tau) = \sum_a \frac{\partial}{\partial x_a} \left[\frac{\partial W}{\partial x_a} + \beta^{-1} \frac{\partial}{\partial x_a} \right] P(x_1, \ldots, x_N, \tau) \qquad \tau > 0 \tag{9}$$

subjected to the initial condition:

$$P(x_1, \ldots, x_N, 0) = \prod_{a=1}^{N} \delta(x_a - x_a^0)$$

where x_a^0 is the initial position of particle a. The stationary solution of the Fokker–Planck equation is

$$P(x_1, \ldots, x_N, \infty) = C_N \mathrm{e}^{-\beta W}. \tag{10}$$

It was shown by Dyson [8] that the time-dependent density

$$\sigma(x, \tau) = \left\langle \sum_a \delta(x - x_a(\tau)) \right\rangle_\tau \tag{11}$$

† More precisely, the density of state–density of state correlation function at different fluxes.

(here $\langle\ldots\rangle_\tau$ denotes an average with 'time-dependent' weight given in equation (9)) satisfies a nonlinear conservation law in the 'hydrodynamical' approximation. The non-equilibrium density $\sigma(x,\tau)$ evolves in τ according to

$$\frac{\partial}{\partial\tau}\sigma(x,\tau) = \frac{\partial}{\partial x}\left(\sigma(x,\tau)\frac{\partial}{\partial x}\Psi\right) \tag{12}$$

where

$$\Psi(x,\tau) = u(x) - \int_K \mathrm{d}y\,\sigma(y,\tau)\ln|x-y| + \left(\frac{1}{\beta}-\frac{1}{2}\right)\ln[\sigma(x,\tau)]. \tag{13}$$

Note that the 'time'-dependent density is normed to N; $\int_K \mathrm{d}x\,\sigma(x,\tau) = N$, where K is the interval on which the levels are supported.

The solution of equation (12) with equation (13) will enable us to determine the parametric (τ) dependence of quantities related to the eigenvalues x_a.

2. Linearization

The stationary density $\sigma(x) := \sigma(x,\infty)$ of the nonlinear diffusion equation satisfies a self-consistent Hückel-type equation

$$u(x) - \int_K \mathrm{d}y\,\sigma(y)\ln|x-y| + \left(\frac{1}{\beta}-\frac{1}{2}\right)\ln[\sigma(x)] = A = \text{constant}. \tag{14}$$

This suggests that for a sufficiently long time period†, we split the non-equilibrium density into an equilibrium part plus a small perturbation $\rho(x,\tau)$:

$$\sigma(x,\tau) = \sigma(x) + \rho(x,\tau) \tag{15}$$

where $\sigma(x)$ is the equilibrium density. Substituting equation (15) into equation (12) and discarding all terms of $\mathrm{O}(\rho^2)$ gives

$$\frac{\partial}{\partial\tau}\rho(x,\tau) = -\frac{\partial}{\partial x}J(x,\tau) \tag{16}$$

where the 'particle' flux is

$$J(x,\tau) := \frac{\sigma(x)}{\gamma}\frac{\partial}{\partial x}\int_K \rho(y,\tau)\ln|x-y|\,\mathrm{d}y. \tag{17}$$

This particle flux requires the entire distribution, ρ, to specify its value at one point, unlike the ordinary diffusion equation. As an example we consider the Gaussian ensembles with $u(x) = x^2$ and $K = (-\sqrt{N}, \sqrt{N})$. In the $N\to\infty$ limit and scaling into the bulk $x \ll N$, where the density is uniform, $\sigma(x) = D = \text{constant}$ then the diffusion equation becomes

$$\frac{\partial}{\partial\tau}\rho(x,\tau) = -\frac{D}{\gamma}\frac{\partial^2}{\partial x^2}\int_{-\infty}^{+\infty}\mathrm{d}y\,\rho(y,\tau)\ln|x-y|. \tag{18}$$

This is converted into

$$\frac{\partial}{\partial\tau}\bar{\rho}_k(\tau) = -\frac{\pi D}{\gamma}|k|\bar{\rho}_k(\tau) \tag{19}$$

via the Fourier transform [7, 8] $\bar{\rho}_k(\tau) = \int_{-\infty}^{+\infty}\mathrm{d}x\,\mathrm{e}^{\mathrm{i}kx}\rho(x,\tau)$ with the solution

$$\bar{\rho}_k(\tau) = \bar{\rho}_k(\tau')\exp\left(-\frac{\pi D}{\gamma}|k|(\tau-\tau')\right) \qquad \tau \geqslant \tau'. \tag{20}$$

† More precisely $D\tau/\gamma L \gg 1$, where L is the interval over which the density extends [8].

From the τ dependence of $\bar{\rho}_k$, we can infer the τ dependence of

$$\mathrm{Corr}(x,y,\tau) := \langle\sigma(x,\tau)\sigma(y,0)\rangle_{\mathrm{eq}} - \langle\sigma(x,\tau)\rangle_{\mathrm{eq}}\langle\sigma(y,0)\rangle_{\mathrm{eq}} \tag{21}$$

$$= \int_{-\infty}^{+\infty}\frac{\mathrm{d}k}{2\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}p}{2\pi}\exp\left(-\mathrm{i}kx-\mathrm{i}py-\frac{\pi D}{\gamma}|k|\tau\right)\mathrm{Corr}(p,k) \tag{22}$$

where

$$\langle\phi\rangle_{\mathrm{eq}} := \left(\prod_a\int_{-\infty}^{+\infty}\mathrm{d}x_a^0\right)P(x_1^0,\ldots,x_N^0)\phi(x_1^0,\ldots,x_N^0)$$

denotes an average over the initial condition. Here $\mathrm{Corr}(p,k)$ is the Fourier transform of the equilibrium density–density correlation function† [1]

$$\mathrm{Corr}(x-y) = -\frac{1}{\pi^2\beta}\frac{1}{(x-y)^2} \qquad x\neq y.$$

$$\mathrm{Corr}(p,k) = \int_{-\infty}^{+\infty}\mathrm{d}x\int_{-\infty}^{+\infty}\mathrm{d}y\,\exp(\mathrm{i}kx+\mathrm{i}py)\mathrm{Corr}(x-y) = \frac{2}{\pi\beta}\delta(k+p)|k|. \tag{23}$$

Therefore,

$$\mathrm{Corr}(x,y,\tau) = \frac{1}{\beta\pi^2}\,\mathrm{Re}\,\frac{1}{[D\pi\tau/\gamma+\mathrm{i}(x-y)]^2}$$

is the universal 'time'-dependent density–density correlation function when τ is measured in units of $\gamma/\pi D$ [6, 7, 9]‡.

3. Hard edge correlations

We shall now focus our attention on the ensembles where the eigenvalues are supported on the right half-line and $\sigma(x)$ has the universal square root singularity at the origin.

$\mathrm{Corr}(x,y,\tau)$ and $\mathrm{Int}(x,y,\tau)$—the twice integrated version of $\mathrm{Corr}(x,y,\tau)$—are derived in section 3. These are applied to determine the parametric dependence of the variance of arbitrary linear statistics given by equations (51) and (54). As an application in section 4, we compute the variance of $\sum_a[1+x_a(\tau)]^{-1}$, which gives the conductance fluctuation of a quasi-one-dimensional disordered system as a function of the external perturbation. According to the Landauer formula the conductance g is $\sum_a[1+x_a]^{-1}$ [2].

Therefore, we have solved in the hydrodynamical approximation the problem posed in [7] for the case where the equilibrium eigenvalue density displays a hard edge singularity $\sigma(x)\sim D/\sqrt{x}$. Since the translational invariance is no longer valid, results obtained in the bulk scaling limit of the Gaussian ensembles (with $u(x)=x^2$) are no longer applicable [12]. For example, the gap formation probability $E_\beta(0,s)$ of the Laguerre ensembles (with $u(x)=x-\alpha\ln x$, $x>0$, $\alpha>-1$) is distinct from that of the Gaussian ensembles [5, 13, 14]. We expect $\mathrm{Corr}(x,y,\tau)$ to have distinct 'time' decay modes from those found in the bulk scaling case§.

† The exact density–density correlation function for the Gaussian ensemble ($\beta=2$ and $x\neq y$) is $-\left[\frac{\sin\pi(x-y)}{\pi(x-y)}\right]^2$ [1]. The hydrodynamical approximation essentially replaces $[\sin\pi(x-y)]^2$ with $\frac{1}{2}$.

‡ Reference [9] gives the exact result for all $\tau\geqslant 0$ and $\beta=2,1,4$.

§ An application of a dimensional argument on equation (18), obtained by scaling into the bulk of the spectrum, shows that the typical distance covered by a diffusing particle over time t is $|x|\sim t$, which is 'faster' than Einstein diffusion $|x|\sim\sqrt{t}$. Looking ahead to equation (24) the same analysis shows that $|x|\sim t^{2/3}$. This suggests that particle transport near the hard edge is intermediate between ballistic motion and classical diffusion.

The diffusion equation now reads

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x}}\mathrm{P}\int_0^\infty \mathrm{d}y\,\frac{\rho(y,t)}{x-y}\right) \tag{24}$$

where $D\tau/\gamma = t$. (We have written $\rho(x,\tau) = \rho(x,t)$.) With the ansatz $\rho(x,t) \propto \rho(x,\lambda)\mathrm{e}^{-\lambda t}$, equation (24) becomes

$$\lambda\rho(x,\lambda) = \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{x}}\int_0^\infty \mathrm{d}y\,\frac{\rho(y,\lambda)}{x-y}\right) \qquad \lambda > 0. \tag{25}$$

The boundary condition on $\rho(x,\lambda)$ is such that the particle flux vanishes at the boundaries $\lim_{x\to 0} J(x,\lambda) = 0$ and $\lim_{x\to\infty} J(x,\lambda) = 0$. The boundary condition at $x = 0$ reads

$$\lim_{x\to 0}\frac{1}{\sqrt{x}}\mathrm{P}\int_0^\infty \frac{\rho(y,\lambda)}{x-y}\,\mathrm{d}y = 0. \tag{26}$$

Since $\mathrm{P}\int_0^\infty \frac{1}{\sqrt{y}(x-y)}\,\mathrm{d}y = 0$, $x > 0$, we find it convenient to write $\rho(x,\lambda)$ as

$$\rho(x,\lambda) = \tilde{\rho}(x,\lambda) + \frac{\mathrm{cst}(\lambda)}{\sqrt{x}} \tag{27}$$

where $\tilde{\rho}(x,\lambda)$ fulfils a stronger condition

$$\int_0^\infty \mathrm{d}y\,\frac{\tilde{\rho}(y,\lambda)}{y} = 0 \tag{28}$$

and where $\mathrm{cst}(\lambda)$ is a function of λ only and is to be determined later. In appendix A we show that the final solution (equation (42)) fulfils the boundary condition (26). The term $\mathrm{cst}(\lambda)/\sqrt{x}$ is similar to the equilibrium solution and we conjecture that the solution for a general hard edge density always has this structure. We now go on to solve equation (25). This can be accomplished by 'un-folding' the half-line onto the real line [15]. With the change of variables $v = \sqrt{y}$, $u = \sqrt{x}$ we have

$$\int_0^\infty \mathrm{d}y\,\frac{\rho(y,\lambda)}{x-y} = \int_0^\infty \mathrm{d}y\,\frac{\tilde{\rho}(y,\lambda)}{x-y} = \int_0^\infty \mathrm{d}v\,\frac{2v\tilde{\rho}(v^2,\lambda)}{u^2-v^2} \tag{29}$$

$$= \int_0^\infty \mathrm{d}v\,\frac{\tilde{\rho}(v^2,\lambda)}{u-v} + \int_0^\infty \mathrm{d}v\,\frac{-\tilde{\rho}(v^2,\lambda)}{u+v}. \tag{30}$$

Now introduce an odd function of v,

$$\tilde{\rho}_1(v,\lambda) \equiv \tilde{\rho}(v^2,\lambda) \qquad v > 0 \tag{31}$$

$$\tilde{\rho}_1(v,\lambda) \equiv -\tilde{\rho}(v^2,\lambda) \qquad v < 0 \tag{32}$$

and find

$$\int_0^\infty \mathrm{d}y\,\frac{\rho(y,\lambda)}{x-y} = \int_0^\infty \mathrm{d}v\,\frac{\tilde{\rho}_1(v,\lambda)}{u-v} + \int_0^\infty \mathrm{d}v\,\frac{\tilde{\rho}_1(-v,\lambda)}{u+v}$$

$$= \int_0^\infty \mathrm{d}v\,\frac{\tilde{\rho}_1(v,\lambda)}{u-v} + \int_{-\infty}^0 \mathrm{d}v\,\frac{\tilde{\rho}_1(v,t)}{u-v} = \int_{-\infty}^{+\infty} \mathrm{d}v\,\frac{\tilde{\rho}_1(v,\lambda)}{u-v}. \tag{33}$$

The diffusion equation (25) together with equations (33) and (27) becomes

$$\frac{\mathrm{cst}(\lambda)}{u} + \lambda\tilde{\rho}_1(u,\lambda) = \frac{1}{2u}\frac{\partial}{\partial u}\left(\frac{1}{u}\int_{-\infty}^{+\infty}\mathrm{d}v\,\frac{\tilde{\rho}_1(v,\lambda)}{u-v}\right). \tag{34}$$

Although equation (34) is derived for $u > 0$ it is also valid for $u < 0$. This can be seen from the fact that $\tilde{\rho}_1$ is an odd function of u. To proceed further we introduce the even function

$$\tilde{\rho}_2(u, \lambda) := \frac{\tilde{\rho}_1(u, \lambda)}{u} \tag{35}$$

and find $\tilde{\rho}_2(u, \lambda)$ satisfies

$$2\lambda\, \mathrm{cst}(\lambda) + 2\lambda u^2 \tilde{\rho}_2(u, \lambda) = \frac{\partial}{\partial u} \int_{-\infty}^{+\infty} \mathrm{d}v\, \frac{\tilde{\rho}_2(v, \lambda)}{u - v}. \tag{36}$$

We have made use of the flux condition at the origin, equation (28),

$$\int_{-\infty}^{+\infty} \mathrm{d}u\, \tilde{\rho}_2(u, \lambda) = 2 \int_0^{\infty} \mathrm{d}u\, \tilde{\rho}_2(u, \lambda) = 0 \tag{37}$$

to arrive at equation (36). This completes the un-folding of the half-line diffusion equation onto the real line. Clearly, equation (36) can be solved by a Fourier transformation.

Equation (37) implies that $\tilde{\rho}_2(u, \lambda)$ is an oscillatory function of u and its Fourier transform

$$\rho_2(k, \lambda) := \int_{-\infty}^{+\infty} \mathrm{d}u\, \mathrm{e}^{\mathrm{i}ku} \tilde{\rho}_2(u, \lambda) \tag{38}$$

vanishes at $k = 0$. A further condition is that ρ_2 is even in k.

Once $\rho_2(k, \lambda)$ is found, the original density can be recovered by the standard inversion formulae

$$\frac{\tilde{\rho}(x, \lambda)}{\sqrt{x}} = \frac{1}{\pi} \int_0^{\infty} \mathrm{d}k\, \cos\left(k\sqrt{x}\right) \rho_2(k, \lambda). \tag{39}$$

A simple calculation shows that the transformed density $\rho_2(k, \lambda)$ satisfies an Airy equation with a point source

$$\frac{\mathrm{d}^2}{\mathrm{d}k^2} \rho_2(k, \lambda) + \frac{\pi |k|}{2\lambda} \rho_2(k, \lambda) = 2\pi\, \mathrm{cst}(\lambda) \delta(k). \tag{40}$$

The solution of this equation is a linear combination of $\mathrm{Ai}(-(\pi/2\lambda)^{1/3}|k|)$ and $\mathrm{Bi}(-(\pi/2\lambda)^{1/3}|k|)$, and reads up to a constant,

$$\begin{aligned} \rho_2(k, \lambda) &= 3^{7/6} \Gamma(2/3) (2\lambda\pi^5)^{1/6} \left[\mathrm{Bi}(0) \mathrm{Ai}\left(-\left(\frac{\pi}{2\lambda}\right)^{1/3} |k| \right) - \mathrm{Ai}(0) \mathrm{Bi}\left(-\left(\frac{\pi}{2\lambda}\right)^{1/3} |k| \right) \right] \\ &= \pi \sqrt{|k|} J_{1/3} \left(\frac{2}{3} \left(\frac{\pi}{2\lambda}\right)^{1/2} |k|^{3/2} \right) \end{aligned} \tag{41}$$

where we have made use of $\mathrm{Ai}(0) = \frac{3^{-2/3}}{\Gamma(2/3)}$, $\mathrm{Bi}(0) = \frac{3^{-1/6}}{\Gamma(2/3)}$ and the relations between $\mathrm{Ai}(-x)$, $\mathrm{Bi}(-x)$ and $J_{\pm\frac{1}{3}}(\frac{2}{3}x^{3/2})$ to arrive at equation (41). Note that $\rho_2(k, \lambda)$ vanishes at $k = 0$ and is even in k. We now use the jump discontinuity of $\frac{\mathrm{d}}{\mathrm{d}k}\rho_2$ across $k = 0$ to determine $\mathrm{cst}(\lambda)$:

$$\mathrm{cst}(\lambda) = \frac{\partial}{\partial k} \left[\sqrt{k} J_{1/3} \left(\frac{2}{3} \sqrt{\frac{\pi}{2\lambda}} |k|^{3/2} \right) \right]_{k=-0}^{k=+0} = \left(\frac{\pi}{18\lambda} \right)^{1/6}.$$

Transforming back to $\rho(x, \lambda)$ with the help of equations (39) and (27) we obtain

$$\rho(x, \lambda) = \sqrt{x} \int_0^{\infty} \mathrm{d}k\, \cos(k\sqrt{x}) \sqrt{k} J_{1/3} \left(\frac{2}{3} \sqrt{\frac{\pi}{2\lambda}} |k|^{3/2} \right) + \frac{(\pi/18\lambda)^{1/6}}{\sqrt{x}}. \tag{42}$$

Therefore the 'time'-dependent density reads as

$$\tilde{\rho}(x,t) = \int_0^\infty \mathrm{d}\lambda\, C(\lambda) \mathrm{e}^{-\lambda t} \rho(x,\lambda) \tag{43}$$

where $C(\lambda)$ is determined by an initial condition.

Equations (42) and (43) will now be used to find the dynamical density–density correlation function. It is clear that $\mathrm{Corr}(x, y, t)$ and

$$\mathrm{ICorr}(x,y,t) := \int_0^y \mathrm{d}z\, \mathrm{Corr}(x,z,t) \tag{44}$$

satisfy the same diffusion equation as $\rho(x,t)$. Hence,

$$\mathrm{ICorr}(x,y,t) = \int_0^\infty \mathrm{d}\lambda\, C(\lambda, y) \mathrm{e}^{-\lambda t} \rho(x,\lambda) \tag{45}$$

and

$$\mathrm{Int}(x,y,t) := \int_0^x \mathrm{d}z\, \mathrm{ICorr}(z,y,t) \tag{46}$$

$$= \pi \int_0^\infty \mathrm{d}\lambda\, C(\lambda, y) \mathrm{e}^{-\lambda t} \int_0^\infty \mathrm{d}k\, \sqrt{k} \sin(k\sqrt{x}) J_{\frac{1}{3}}\left(\frac{2}{3}\left(\frac{\pi}{2\lambda}\right)^{1/2} k^{3/2}\right) \tag{47}$$

where we have used equation (A15) in appendix A to arrive at the last equation. $C(\lambda, y)$ is determined by the 'initial condition'

$$\mathrm{Int}(x,y,t=0) = \frac{1}{\beta\pi^2} \ln\left[\frac{|\sqrt{x}-\sqrt{y}|}{\sqrt{x}+\sqrt{y}}\right].$$

A derivation of this formula can be found in appendix B. To find $C(\lambda, y)$ we first perform a Fourier sine transform in the variable $\sqrt{x}$; $\int_0^\infty \mathrm{d}\sqrt{x} \sin(p\sqrt{x}) \ldots$, on equation (47) at $t = 0$ to obtain an integral equation satisfied by $C(\lambda, y)$†:

$$-\frac{1}{\beta\pi\, p} \sin\left(p\sqrt{y}\right) = \frac{\pi^2}{2} \int_0^\infty \mathrm{d}\lambda\, C(\lambda, y) \sqrt{p} J_{1/3}\left(\frac{2}{3}\sqrt{\frac{\pi}{2\lambda}} p^{3/2}\right). \tag{48}$$

This can be rewritten as

$$-\frac{2\sin(p\sqrt{y})}{\beta\pi^4 p^{3/2}} = \mathcal{H}_{1/3}\left(\frac{C(\pi/2u^2, y)}{u^4}, \tfrac{2}{3}p^{3/2}\right) \tag{49}$$

where $\mathcal{H}_\nu(f(u),\xi) = \bar{f}_\nu(\xi) := \int_0^\infty \mathrm{d}u\, f(u) u J_\nu(u\xi)$ denotes the Hankel transform. Using the Hankel inversion theorem ($f(x) = \int_0^\infty \mathrm{d}\xi\, \bar{f}_\nu(\xi)\xi J_\nu(\xi u)$ [16]), we find

$$\frac{C(\pi/2u^2, y)}{u^4} = -\frac{4}{3\beta\pi^4} \int_0^\infty \mathrm{d}p\, \sin(p\sqrt{y})\sqrt{p} J_{1/3}(\tfrac{2}{3}p^{3/2}u). \tag{50}$$

Therefore, the 'time' dependence of the twice-integrated correlation function is displayed as a Laplace transform with the spectral parameter $\lambda := \pi/2u^2$:

$$\mathrm{Int}(x,y,t) = -\frac{1}{3\beta\pi^2} \int_0^\infty \frac{\mathrm{d}\lambda}{\lambda^2} \mathrm{e}^{-\lambda t} F(x,\lambda) F(y,\lambda) \tag{51}$$

where $F(x,\lambda) := \int_0^x \mathrm{d}z\, \tilde{\rho}(z,\lambda)$ and $\tilde{\rho}(z,\lambda)$ is given by equations (27) and (42). Thus, the variance of an arbitrary linear statistic:

$$\mathcal{Q}(t) := \sum_a \mathcal{Q}(x_a(t)) = \int_0^\infty \mathrm{d}x \sum_a \delta(x - x_a(t)) \mathcal{Q}(x) \tag{52}$$

† We use the transform $\int_0^\infty \mathrm{d}x\, \ln|\frac{x+a}{x-a}| \sin(px) = \frac{\pi}{p}\sin(pa)$.

is

$$\begin{aligned}\mathrm{Var}[Q,t] &:= \langle Q(t)Q(0)\rangle_{eq} - \langle Q(t)\rangle_{eq}\langle Q(0)\rangle_{eq} \qquad (53)\\ &= \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\, Q(x)Q(y)\mathrm{Corr}(x,y,t)\\ &= \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\, Q'(x)Q'(y)\mathrm{Int}(x,y,t). \qquad (54)\end{aligned}$$

This is the main result of our paper.

4. Variance of $\frac{1}{1+x}$

As an example, we take $Q(x) = \frac{1}{1+x}$, which is the Landauer formula for the conductance. A simple calculation gives

$$\mathrm{Var}\left[\frac{1}{1+x}, t\right] = \frac{\pi}{3\beta}\int_0^\infty \frac{\mathrm{d}\lambda}{\lambda^2}\mathrm{e}^{-\lambda t} h^2(\lambda) \qquad (55)$$

with

$$\begin{aligned} h(\lambda) &:= \frac{1}{\pi^2}\int_0^\infty \frac{\mathrm{d}x}{(1+x)^2}F(x,\lambda)\\ &= \frac{1}{2}\int_0^\infty \mathrm{d}k\, k^{3/2}\mathrm{e}^{-k} J_{\frac{1}{3}}\left(\frac{2}{3}\left(\frac{\pi}{2\lambda}\right)^{1/2} k^{3/2}\right)\\ &= \left(\frac{2\lambda}{\pi}\right)^{5/6}\sum_{n=0}^{\infty} A_n \left(\frac{18\lambda}{\pi}\right)^{n/3} \qquad (56)\end{aligned}$$

where

$$A_n := \frac{(-1)^n}{n!}\frac{\Gamma(1+n/3)}{\Gamma(-(n-1)/3)}. \qquad (57)$$

Equations (56) and (57) can be found in [17]. Now when $n \equiv 1 \bmod 3$, A_n is taken to be zero; the remaining terms can be grouped into two sums with $n \equiv 0 \bmod 3$ and $n \equiv 2 \bmod 3$, respectively. An application of the ratio test to these series shows that they both converge absolutely and uniformly for all λ. We deduce that (with $\beta = 2$)

$$\frac{\pi}{6}\frac{h^2(\lambda)}{\lambda} = \lambda^{2/3}\sum_{n=0}^{\infty} B_n \lambda^{n/3} \qquad (58)$$

converges uniformly in the $\lambda^{1/3}$ plane. Using this representation and after performing the Laplace transform we obtain

$$\mathrm{Var}\left[\frac{1}{1+x}, t\right] = \frac{1}{t^{2/3}}\sum_{n=0}^{\infty} B_n \frac{\Gamma(n+2/3)}{t^{n/3}} = \frac{C_{2/3}}{t^{2/3}} + \frac{C_{4/3}}{t^{4/3}} + \frac{C_{5/3}}{t^{5/3}} + \frac{C_2}{t^2} + \frac{C_{7/3}}{t^{7/3}} + \cdots \qquad (59)$$

where

$$\begin{aligned} C_{2/3} &= \frac{(\frac{3}{2\pi^2})^{1/3}\Gamma(2/3)}{2\Gamma^2(1/3)} \approx 0.050\,346\\ C_{4/3} &= \frac{3(\frac{3}{2\pi^2})^{2/3}\Gamma(5/3)\Gamma(4/3)}{\Gamma(-1/3)\Gamma(1/3)} \approx -0.063\,287\,1\end{aligned}$$

$$C_{5/3} = -\frac{3(\frac{3}{2\pi^5})^{1/3}\Gamma(5/3)}{\Gamma(1/3)\Gamma(-2/3)} \approx 0.042735$$

$$C_2 = \frac{27\Gamma^2(5/3)}{4\pi^2\Gamma^2(-1/3)} \approx 0.0337737$$

$$C_{7/3} = -\Gamma(7/3)\left[\frac{9(\frac{9}{4\pi^7})^{1/3}\Gamma(5/3)}{\Gamma(-1/3)\Gamma(-2/3)} - \frac{9(\frac{9}{4\pi^7})^{1/3}\Gamma(8/3)}{10\Gamma(-4/3)\Gamma(1/3)}\right] \approx -0.0716263.$$

This completes the determination of the long 'time' behaviour of $\mathrm{Var}[\frac{1}{1+x}, t]$. According to the Watson lemma [18], equation (59) is an asymptotic expansion. The fact that the Brownian-motion ensemble tends to the stationary ensemble with exponential speed as soon as $\tau D/\gamma L \gg 1$, enables us to conclude that $\lim_{t\to\infty} \mathrm{Var}[\mathcal{Q}, t] = 0$. At $t = 0$ we simply go back to equation (55) and make use of the completeness theorem of the Hankel transform to show that $\mathrm{Var}[\frac{1}{1+x}, 0] = \frac{1}{8\beta}$, a result obtained previously [3, 19, 20]. Other examples of $\mathcal{Q}$ can be found in [19].

Appendix A. A check of the solution

In this appendix we are going to show that

$$\rho(x,\lambda) = \frac{(\pi/18\lambda)^{1/6}}{\sqrt{x}} + \int_0^\infty \sqrt{x}\cos(k\sqrt{x})\sqrt{k}J_{1/3}\left(\frac{2}{3}\sqrt{\frac{\pi}{2\lambda}}|k|^{3/2}\right)\mathrm{d}k \quad \text{(A1)}$$

is a solution to the integro-differential equation

$$\lambda\rho(x,\lambda) = \frac{\partial}{\partial x}\frac{1}{\sqrt{x}}\int_0^\infty \frac{\rho(y,\lambda)}{x-y}\,\mathrm{d}y. \quad \text{(A2)}$$

To obtain better convergence in the manipulations we will instead show $\rho(x,\lambda)$ fulfils the equation obtained by integrating equation (A2) from 0 to z. From the boundary condition equation (26), we see that the lower boundary term on the right-hand side will vanish; then equation (A2) will become

$$\lambda\int_0^z \mathrm{d}x\,\rho(x,\lambda) = \frac{1}{\sqrt{z}}\int_0^\infty \frac{\rho(y,\lambda)}{z-y}\,\mathrm{d}y. \quad \text{(A3)}$$

This equation is a stronger statement than equation (A2) since any constant will vanish when equation (A3) is differentiated to give equation (A2). We start by evaluating the right-hand side of equation (A3):

$$\begin{aligned}
\mathrm{RHS} &= \frac{1}{\sqrt{z}}\int_0^\infty \frac{\rho(y,\lambda)}{z-y}\,\mathrm{d}y \quad &\text{(A4)}\\
&= \frac{1}{\sqrt{z}}\int_0^\infty \frac{(\pi/18\lambda)^{1/6}}{z-y}\frac{\mathrm{d}y}{\sqrt{y}}\\
&\quad +\frac{1}{\sqrt{z}}\int_0^\infty \mathrm{d}y\int_0^\infty \frac{\sqrt{y}\cos(k\sqrt{y})}{z-y}\sqrt{k}J_{1/3}\left(\frac{2}{3}\sqrt{\frac{\pi}{2\lambda}}|k|^{3/2}\right)\mathrm{d}k\\
&= \frac{1}{\sqrt{z}}\int_0^\infty \mathrm{d}y\int_0^\infty \frac{\sqrt{y}\cos(k\sqrt{y})}{z-y}\sqrt{k}J_{1/3}\left(\frac{2}{3}\sqrt{\frac{\pi}{2\lambda}}|k|^{3/2}\right)\mathrm{d}k. \quad &\text{(A5)}
\end{aligned}$$

Using the identity

$$\int_0^\infty \frac{\sqrt{y}\cos(ky)}{z-y}\,\mathrm{d}y = 2\pi\delta(k) + \pi\sqrt{z}\sin(k\sqrt{z})$$

in equation (A5) gives

$$\frac{1}{\sqrt{z}}\int_0^\infty \frac{\rho(y,\lambda)}{z-y}\,\mathrm{d}y = \pi\int_0^\infty \sin(k\sqrt{z})\sqrt{k}J_{1/3}\left(\frac{2}{3}\sqrt{\frac{\pi}{2\lambda}}|k|^{3/2}\right)\mathrm{d}k. \tag{A6}$$

We now turn to the left-hand side of equation (A3):

$$\mathrm{LHS} = \lambda\int_0^z \mathrm{d}x\,\rho(x,\lambda) \tag{A7}$$

$$\begin{aligned} &= \lambda\int_0^z \frac{(\pi/18\lambda)^{1/6}\,\mathrm{d}x}{\sqrt{x}} + \lambda\int_0^z \mathrm{d}x\int_0^\infty \sqrt{x}\cos(k\sqrt{x})\sqrt{k}J_{1/3}\left(\frac{2}{3}\sqrt{\frac{\pi}{2\lambda}}|k|^{3/2}\right)\mathrm{d}k \\ &= 2\lambda\left(\frac{\pi}{18\lambda}\right)^{1/6}\sqrt{z} + 2\lambda\int_0^\infty\int_0^{\sqrt{z}} \mathrm{d}t\,t^2\cos(k\,t)\sqrt{k}J_{1/3}\left(\frac{2}{3}\sqrt{\frac{\pi}{2\lambda}}|k|^{3/2}\right)\mathrm{d}k. \end{aligned} \tag{A8}$$

The integral with respect to $t=\sqrt{x}$ is

$$\int_0^{\sqrt{z}} \mathrm{d}t\,t^2\cos(kt) = -\frac{\partial^2}{\partial k^2}\left\{\frac{\sin(k\sqrt{z})}{k}\right\}. \tag{A9}$$

The left-hand side is then

$$\begin{aligned} \mathrm{LHS} &= 2\lambda\left(\frac{\pi}{18\lambda}\right)^{1/6}\sqrt{z} - 2\lambda\int_0^\infty \frac{\partial^2}{\partial k^2}\left\{\frac{\sin(k\sqrt{z})}{k}\right\}\sqrt{k}J_{1/3}\left(\frac{2}{3}\sqrt{\frac{\pi}{2\lambda}}|k|^{3/2}\right)\mathrm{d}k \\ &= 2\lambda\left(\frac{\pi}{18\lambda}\right)^{1/6}\sqrt{z} - 2\lambda\left[\frac{\partial}{\partial k}\left\{\frac{\sin(k\sqrt{z})}{k}\right\}\sqrt{k}J_{1/3}\left(\frac{2}{3}\sqrt{\frac{\pi}{2\lambda}}|k|^{3/2}\right)\right]_0^\infty \\ &\quad +2\lambda\int_0^\infty \frac{\partial}{\partial k}\left\{\frac{\sin(k\sqrt{z})}{k}\right\}\frac{\partial}{\partial k}\left\{\sqrt{k}J_{1/3}\left(\frac{2}{3}\sqrt{\frac{\pi}{2\lambda}}|k|^{3/2}\right)\right\}\mathrm{d}k. \end{aligned} \tag{A10}$$

The boundary terms from the partial integration are zero since

$$\begin{aligned} \sqrt{k}J_{1/3}\left(\frac{2}{3}\sqrt{\frac{\pi}{2\lambda}}|k|^{3/2}\right) &\sim \text{constant } k^{-1/4}\cos(\text{constant } k^{3/2} - \text{phase}) \qquad k\to\infty \\ &= 0 \qquad k=0. \end{aligned} \tag{A11}$$

Now

$$\begin{aligned} \mathrm{LHS} &= 2\lambda\left(\frac{\pi}{18\lambda}\right)^{1/6}\sqrt{z} + 2\lambda\left[\frac{\sin(k\sqrt{z})}{k}\frac{\partial}{\partial k}\left\{\sqrt{k}J_{1/3}\left(\frac{2}{3}\sqrt{\frac{\pi}{2\lambda}}|k|^{3/2}\right)\right\}\right]_0^\infty \\ &\quad -2\lambda\int_0^\infty \frac{\sin(k\sqrt{z})}{k}\frac{\partial^2}{\partial k^2}\left\{\sqrt{k}J_{1/3}\left(\frac{2}{3}\sqrt{\frac{\pi}{2\lambda}}|k|^{3/2}\right)\right\}\mathrm{d}k. \end{aligned} \tag{A12}$$

The boundary term at infinity vanishes; the boundary term at the origin is

$$-2\lambda\sqrt{z}\frac{\partial}{\partial k}\left\{\sqrt{k}J_{1/3}\left(\frac{2}{3}\sqrt{\frac{\pi}{2\lambda}}|k|^{3/2}\right)\right\}\bigg|_{k=0} = -2\lambda\left(\frac{\pi}{18\lambda}\right)^{1/6}\sqrt{z}. \tag{A13}$$

By differentiation

$$\frac{1}{k}\frac{\partial^2}{\partial k^2}\left\{\sqrt{k}J_{1/3}\left(\frac{2}{3}\sqrt{\frac{\pi}{2\lambda}}|k|^{3/2}\right)\right\} = -\frac{\pi}{2\lambda}\sqrt{k}J_{1/3}\left(\frac{2}{3}\sqrt{\frac{\pi}{2\lambda}}|k|^{3/2}\right). \tag{A14}$$

Gathering the pieces we have

$$\lambda\int_0^z \mathrm{d}x\,\rho(x,\lambda) = \pi\int_0^\infty \sin(k\sqrt{z})\sqrt{k}J_{1/3}\left(\frac{2}{3}\sqrt{\frac{\pi}{2\lambda}}|k|^{3/2}\right)\mathrm{d}k \tag{A15}$$

and

$$\lambda \int_0^z \mathrm{d}x\, \rho(x,\lambda) = \frac{1}{\sqrt{z}} \int_0^\infty \frac{\rho(y,\lambda)}{z-y}\,\mathrm{d}y. \tag{A16}$$

Here we make two remarks on the above derivations.

(i) Since the number of particles in the interval $(0, z)$ goes to zero, when z tends to zero the last equation shows that the zero flux condition at the origin is fulfilled.

(ii) This concerns the $z \to \infty$ limit in equation (A15). The integrand on the right-hand side is the product of two out-of-phase strongly oscillating functions. Since neither of them are absolutely integrable, a Riemann–Lebesque lemma cannot be used to conclude that the integral tends to zero. However, we may appeal to the theory of generalized functions and state that it is zero. The RHS of equation (A15) can be rewritten up to irrelevant constants as

$$\mathcal{J}(a) = \int_0^\infty \mathrm{d}y\, \sin(ya)\sqrt{y}J_{1/3}(by^{3/2})$$

where $a := \sqrt{z}$ and $b := \frac{2}{3}(\pi/2\lambda)^{1/2}$. We wish to determine $\mathcal{J}(a)$ in the limit $a \to \infty$ with b (> 0) fixed. Consider instead,

$$\mathcal{J}_\mu(a) = \int_0^\infty \mathrm{d}y\, \frac{\sin(ya)}{y} y^{1-\mu} J_{1/3}(by^{3/2}) \qquad -\tfrac{5}{4} < \operatorname{Re}\mu < \tfrac{5}{2}.$$

The integral we need can be defined as the analytic continuation of $\mathcal{J}_\mu(a)$ to $\mu = -\frac{1}{2}$. Now, since $\lim_{a\to\infty} \sin(ax)/x = \pi\delta(x)$ we conclude, by integrating over the δ function, that $\lim_{a\to\infty} \mathcal{J}_\mu(a) = 0$. Hence $\mathcal{J}(\infty) = 0$. Physically this means that the total number of particles in each lambda mode is zero. The zero mode, of course, contains particles.

Appendix B. The equal time variance of a linear statistic

The equal time variance of a linear statistic $\mathcal{Q}$ is

$$\mathrm{Var}(\mathcal{Q},0) = \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\, \mathcal{Q}(x)\mathcal{Q}(y)\,\mathrm{Corr}(x,y,0) \tag{B17}$$

a result first derived in [19] by the method of functional derivatives. The claim in [19] is that the method is valid for all potentials; however, this is flawed by the use of the $N = \infty$ density which may not exist for *all* potentials. For the sake of completeness, we present in this appendix a small modification of the arguments given in [19] and show that the result has a more general validity. Instead of focusing on equation (B17), we prefer to work with the partially integrated expressions

$$\mathrm{Var}(\mathcal{Q},t) = -\int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\, \mathcal{Q}(x)\mathcal{Q}'(y)\,\mathrm{ICorr}(x,y,t) \tag{B18}$$

and

$$\mathrm{Var}(\mathcal{Q},t) = \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\, \mathcal{Q}'(x)\mathcal{Q}'(y)\,\mathrm{Int}(x,y,t) \tag{B19}$$

where†

$$\mathrm{ICorr}(x,y,t) \equiv \int_0^y \mathrm{d}z\, \mathrm{Corr}(x,z,t) \tag{B20}$$

$$\mathrm{Int}(x,y,t) \equiv \int_0^x \mathrm{d}z\, \mathrm{ICorr}(z,y,t) = \int_0^x \mathrm{d}z_1 \int_0^y \mathrm{d}z_2\, \mathrm{Corr}(z_1,z_2,t). \tag{B21}$$

† To avoid the boundary terms it has been assumed that the linear statistic fulfils $\sqrt{x}\,\mathcal{Q}(x)|_{x=0} = 0$. The linear statistic corresponding to the conductance $\frac{1}{1+x}$ satisfies this criterion.

We start by finding $\mathrm{ICorr}(x, y, 0)$. By definition $\sigma_{\mathrm{gas}}(x) = \sum_{n=1}^{\infty} \delta(x - x_n^0)$ and

$$\langle \sigma_{\mathrm{gas}}(x) \rangle_{\mathrm{eq}} = \frac{\int_0^\infty \mathrm{d}x_1^0 \ldots \int_0^\infty \mathrm{d}x_N^0\, \sigma_{\mathrm{gas}}(x) \mathrm{e}^{-W[u]}}{\int_0^\infty \mathrm{d}x_1^0 \ldots \int_0^\infty \mathrm{d}x_N^0\, \mathrm{e}^{-W[u]}}. \tag{B22}$$

The dependence of the energy W on the coordinates $x_1^0, x_2^0, \ldots, x_N^0$ has been suppressed. The external potential $u(x)$ is assumed to be bounded at the origin which is the physically interesting situation. Now write $u(x) = -\int_0^x f(z)\,\mathrm{d}z + u(0)$, where $f(x)$ is the force. The constant $u(0)$ can be set to zero as this can always be accomplished by a redefinition of the zero-point energy. The energy is now

$$W[u] = W\left[-\int_0^x f(z)\,\mathrm{d}z\right] = -\sum_{n=1}^{N} \int_0^\infty f(z)\theta(x_n^0 - z)\,\mathrm{d}z - \sum_{i<j} \ln |x_j^0 - x_i^0| \tag{B23}$$

where θ is the Heaviside step function. The functional derivative of $W[-\int_0^x f(z)\,\mathrm{d}z]$ with respect to f is

$$\frac{\delta W[-\int_0^x f(z)\,\mathrm{d}z]}{\delta f(y)} = -\sum_{n=1}^{\infty} \theta(x_n^0 - y) = -\int_0^y \sigma_{\mathrm{gas}}(x, 0)\,\mathrm{d}z. \tag{B24}$$

The functional derivative of $\sigma_{\mathrm{eq}}(x) = \langle \sigma_{\mathrm{gas}}(x, 0) \rangle_{\mathrm{eq}}$ with respect to f is recognized as $\mathrm{ICorr}(x, y, 0)$:

$$\mathrm{ICorr}(x, y, 0) = \frac{\delta \sigma_{\mathrm{eq}}(x)}{\beta \delta f(y)}. \tag{B25}$$

The density $\sigma_{\mathrm{eq}}(x)$ is approximated by

$$\sigma(x, N; f) = -\frac{1}{\pi^2} \sqrt{\frac{b(N, f) - x}{x}} \int_0^{b(N,f)} \frac{\mathrm{d}z}{z - x} \sqrt{\frac{z}{b(N, f) - z}} f(z) \tag{B26}$$

where $b(N, f)$ is the upper limit of support of $\sigma(x, N; f)$. When $\epsilon\delta(x - y)$ is added to the force $f(x)$, $[\delta\sigma(x, N; f)]/[\delta f(y)]$ is computed as the linear term in ϵ. We have

$$\sigma(x, N; f + \epsilon\delta(x - y)) = -\frac{1}{\pi^2} \sqrt{\frac{b - x}{x}} \int_0^b \frac{\mathrm{d}z}{z - x} \sqrt{\frac{z}{b - z}} [f(z) + \epsilon\delta(z - y)] \tag{B27}$$

$$= \epsilon \frac{1}{\pi^2} \sqrt{\frac{y}{x}} \frac{1}{x - y} \sqrt{\frac{b - x}{b - y}} + \sigma(x, N - \eta; f) \tag{B28}$$

where

$$\eta = \eta(\epsilon, y, N, f) = \epsilon \frac{1}{\pi^2} \int_0^b \sqrt{\frac{y}{x}} \frac{1}{x - y} \sqrt{\frac{b - x}{b - y}}\,\mathrm{d}x = -\epsilon \frac{1}{\pi} \frac{y}{b - y}$$

is the number of particles associated with the force $\epsilon\delta(x - y)$ and $b := b(N, f + \epsilon\delta(x - y)) = b(N - \eta, f)$ is the upper limit of support of $\sigma(x; N, f + \epsilon\delta(x - y))$ and $\sigma(x, N - \eta; f)$†.

† They have the same upper limit because the part of $\sigma(x, N; f + \epsilon\delta(x - y))$ corresponding to f:

$$-\frac{1}{\pi^2} \sqrt{\frac{b - x}{x}} \int_0^b \frac{\mathrm{d}z}{z - x} \sqrt{\frac{z}{b - z}} f(z)$$

is a solution, with $N - \eta$ particles, to the problem with the external force f obeying the right boundary conditions. In other words

$$-\frac{1}{\pi^2} \sqrt{\frac{b - x}{x}} \int_0^b \frac{\mathrm{d}z}{z - x} \sqrt{\frac{z}{b - z}} f(z) = \sigma(x, N - \eta; f).$$

The functional derivative is

$$\frac{\delta\sigma(x,N)}{\beta\delta f(y)} = \frac{1}{\beta\pi^2}\sqrt{\frac{y}{x}}\frac{1}{x-y}\frac{b-x}{b-y} + \frac{1}{\beta\pi}\frac{y}{b-y}\frac{\partial\sigma(x,N)}{\partial N}. \tag{B29}$$

In the limit $N \to \infty$, $\sqrt{(b-x)/(b-y)} = 1$, η and $\partial\sigma(x,N)/\partial N$ tend to zero. The extra density $\Delta\sigma(x)$ when ϵ particles are added to the system does not experience any force in the interval $(0,b)$. Therefore, $\Delta\sigma(x)$ is less than the corresponding density with ϵ particles in the interval $(0,b)$:

$$\frac{1}{\pi}\frac{\epsilon}{\sqrt{x(b-x)}} \qquad x \in (0,b).$$

Outside the interval $(0,b)$, $\Delta\sigma(x)$ tends monotonically to zero. We conclude, in this limit, that the functional derivative is reduced to

$$\frac{\delta\sigma(x,N)}{\beta\delta f(y)} = \frac{1}{\beta\pi^2}\sqrt{\frac{y}{x}}\frac{1}{x-y} \tag{B30}$$

$$\mathrm{ICorr}(x,y,0) = \frac{1}{\beta\pi^2}\frac{\partial}{\partial x}\ln\left|\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right|. \tag{B31}$$

From this it is seen that

$$\mathrm{Int}(x,y,0) = \frac{1}{\beta\pi^2}\ln\left|\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right| \tag{B32}$$

and

$$\mathrm{Var}(\mathcal{Q},0) = \frac{1}{\beta\pi^2}\int_0^\infty \mathrm{d}x\int_0^\infty \mathrm{d}y\, \mathcal{Q}'(x)\mathcal{Q}'(y)\ln\left|\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right|. \tag{B33}$$

Thus at equal time, variances are independent of the potential u.

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where $G(.)$ is the Barnes G-function.

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